

The Improvement of Mathematical
Intuition: Literature Review

Master's Project

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by

Mansureh Baghadoost
School of Education
University of Dayton
Dayton, Ohio
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Approved by:

Official Advisor

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CHAPTER I

Introduction

It takes much time and effort to become a mathematical teacher. It is a never-ending process. Each individual is influenced by many, but every once in awhile we study or work with someone who exerts a major role upon the way we shall go. Teachers are the most important educational influences on students' learning of mathematics. From Kindergarten to high school, students spend thousands of hours in direct contact with teachers. While other educational agents may have influence on educational decisions, it is the day by day contact with teachers which is the main influence of the formal educational institution(National Council Association of Teachers of Mathematics, 1982).

Many people believe that teacher's attitudes toward mathematics are important determinants of pupil's interest in mathematics. Studies by NCATM shows that some students have high aptitude for school mathematics and some are poor mathematics learners. The first group routinely attempted to interpret the symbolic structures and rules they are taught in terms of quantities and relationships to which the symbols

refer, whereas the second group try to learn mathematics as pure symbol manipulation .Good mathematics learners expect to be able to make sense of the rules they are taught, and they apply some energy and time to the task of making sense . By contrast, those less adept in mathematics try to memorize and apply the rules that are thought, but do not attempt to relate these rules to what they know about mathematics at a more intuitive level.

The question of how to teach mathematics effectively is not an easy one to answer. The main focus of this project is to reveal developing mathematical thinking and reasoning abilities in young school children by reviewing literature..

Significance of the Problem - For many years now, most efforts to improve educational outcomes for disadvantaged students have been based on the premise that what such children need is higher expectations for learning coupled with intensified and careful application of traditional classroom methods. Thus, what is typically prescribed is more careful explanations, more practice, and more frequent testing to monitor progress. Such methods seem to work up to a point. That is, they produce gains on skills tests, but

they are not designed to teach children to reason and solve problems today. Such abilities are fundamental for participation in the economy and society in general.

The nearly exclusive focus on the kinds of "basic skills" that can be taught by repetitive drill does not necessarily derive from a lack of ambition for disadvantaged students or from a belief that the children are inherently incapable of thinking and problem solving. Rather, it is rooted in an assumption that most educators share about all learning by nearly all children (some would except the "gifted"): that successful learning means working step by step through a hierarchical sequence of skills and concepts. The common view is that skills and concepts are ordered in rather strict hierarchies and that asking children to perform complex skills before they master the prerequisite simpler ones is to doom them to failure, or at least to frustration, in the course of learning. This hierarchical mastery learning approach dictates that children who have trouble learning some of the simpler skills practice them longer. But in practice this turns out to deny disadvantaged children the opportunity to learn higher order abilities. Because many disadvantaged are among those who learn slowly at the outset, they are doomed to more and more supervised

practice on the "basics". They never get to graduate to the more demanding and interesting problems that constitute the "higher order" part of the curriculum.

PURPOSE - The purpose of this project is to argue that disadvantaged children, like all children, can begin their educational life by engaging in active thinking and problem solving. It argues further that when thinking oriented instruction is carefully organized for this purpose, children can acquire the traditional basic skills in the process of reasoning and solving problems. As a result, they can acquire not only the fundamentals of a discipline but also the ability to apply those fundamentals, and critically a belief in their own capacities as learners and thinkers.

CHAPTER II

Review Of The Literature

Research on children's knowledge and learning of mathematics has been one of the most active topics in developmental cognitive psychology in recent years. The results have been not only an explosion of research studies but also a significant reconceptualization of the nature of early mathematical knowledge, of how children acquire such knowledge informally, and of how mathematics learning proceeds in school. Relevant research has been conducted by cognitive, developmental, and educational psychologists as well as by a vibrant community of mathematics educators. Despite their diverse training and affiliation, there is broad agreement among these various research groups on what can be termed a constructivist assumption about how mathematics is learned. It is assumed that mathematical knowledge like all knowledge is not directly absorbed but is constructed by each individual. This constructivist view is consonant with the theory of Jean Piaget but comes in many varieties and does not necessarily imply either a stage theory or the logical determinism of orthodox Piagetian theory.

The work presented here is to review research on learning and cognition. Further more, how reasoning and thinking abilities develop and might be cultivated in arithmetic class.

How should teachers teach? Before any one can answer this question another question must be asked: How do students learn? The art of teaching, if it is to be effective, must be based on an adequate theory of learning.

There have been many different theories of learning which can be classified according to their views of the relation between the child and his environment.

Piaget - One of the most important theories of learning is the developmental psychology of Jean Piaget. It is based in part on studies of how the child develops his conceptions of number and space. Piaget finds that the child passes through four distinct stages of mental growth, which he calls the sensori motor stage, the pre-operational stage, the stage of concrete operations, and the stage of formal operations (Ginsburg Herbert, and Sylvia Oppen, 1969)

Since The child's mental growth advances through qualitatively distinct stages, these stages should be taken into account when we plan the curriculum.

Piaget emphasizes two things about active learning. First a child must be allowed to do things over and over again and thus reassure himself that what he has learned is true. Secondly, this practice should be enjoyable(Ginsburg Herbert, and Sylvia Oppen, 1969).

The conclusion from studies of child development theory is that two criteria must be considered for selecting the mathematical experiences that a child should be taught at any given age: (a) They should be experiences that he or she is ready for, in view of the stage of mental growth that the child has reached. (b) They should prepare the child to advance to the next stage. The right timing for teaching a topic is essential. A topic should not be taught too early, but neither delayed for years once the child is ready for learning it.

Bruner - Bruner, a psychologist at Harvard University has also done some exceptional studies concerning the learning process(Craig Grace, 1979). Bruner's work

explains how learning takes place, starting from the same focal point as Piaget. He also pays attention to the improvement of learning rather than its description. As he examined the "act of learning", Bruner finds that three phases seem to be occurring almost simultaneously. He describes them as follows: First there is acquisition of new information - often information that runs counter to, or is a replacement for what the person has previously known. It is a refinement of previous knowledge. A second aspect of learning may be called transformation: the process of manipulating knowledge to make it fit new tasks, the way we deal with information in order to go beyond it. A third aspect of learning is evaluation: checking whether the way we have manipulated information is adequate to the task.

Where as Piaget has tried to explain what the child is capable of learning at a given stage of his life, Bruner proceeds to describe the action undertaken to get there. Bruner has based his analysis on the startling hypothesis that any child at any stage of development can be taught any subject in some honest form! This would seem to be a direct contradiction of Piaget's conclusion that a child in a particular learning stage can not master certain basic concepts (Craign Grace,1979).

If Bruner's hypothesis is granted, it would seem that it would no longer be necessary to consider whether or not the student could learn a given concept. Instead, curriculum designers only need determine what is important to include and then construct an acceptable form of the content. This would be a major change in points of emphasis (Craign Grace,1979). Teachers need no longer ask "Can my students learn this?" They would, instead focus upon, "How can I organize these ideas so that my students will find them accessible?"

Bruner emphasizes that individuals translate their experience into their own models of the world - each person's reality. He spells out three points of emphasis in this development, moving from active manipulation of objects, to perceptual organization and imagery, and then to symbolic representation using language or words. Except for his assumption that anyone can be taught anything in some honest form, Bruner seems to be in general agreement with Piaget.

Sawyer - W.W.Sawyer is a mathematician and not a psychologist. He has been concerned with helping children to learn mathematics and has formulated some of his own ideas on the subject that are so simple and direct as to deserve brief consideration here. Sawyer

observes that there is a widespread fallacy among teachers that memorization is easy and understanding is difficult. He states emphatically that the opposite is true because it is very hard to remember that which you do not understand. To achieve understanding, Sawyer feels that it is important for a child to have a visual image of the mathematical idea under consideration - a visual anchor, so to speak, for the abstraction (Craig Grace, 1979).

Many other studies have been undertaken to try to expose the learning process to view. Those and the work of Piaget, Bruner and Sawyer have arrived at differing points of view regarding the learning process but they have also established common characteristics as well. The teacher who is intent upon building mathematical experiences for children can not ignore these works if he is to help each of his students to experience the joys and satisfaction that are all too often missing in the mathematics classroom. To aid the teacher in developing this, Sawyer has created a great collection of visual representations. Thus, he has translated the general goal of achieving understanding into a specific process of aiding visualization because he sees visualization as a key to understanding.

Neo-Piagetian Theories - Virtually all psychologists of cognition, whether they come from an individual differences, a developmental, or an information processing perspective, share the view that it is essential to try to identify an individual's thinking and reasoning competencies independently of their performances in any particular occasion.

Information processing psychology has shared the view that thinking abilities are mental capacities that are owned by the individual, without reference to conditions of use. Early similar work on the cognitive processes entailed in problem solving (Newell & Simon, 1972) aimed to uncover the exact processes used in solving particular problems. In an effort to link information processing with individual difference research, the tools and concepts of information processing psychology were extended during the 1970s and early 1980s to cognitive analyses of performance on ability and aptitude tests (Pellegrino & Glaser, 1982; Sternberg, 1977). Processes identified in these analyses subsequently became the target of direct instruction in courses of generalized problem solving skills and higher order thinking (Sternberg 1986). However, as Resnick (1987 a) pointed out in an analysis of the prospects for teaching higher order thinking skills, although there

have been successes in raising ability test scores as a result of such training, there is no evidence that people then apply the taught abilities to real world or school learning situations. Recent advances in research in thinking and problem solving in various domains of subject matter learning and technical performance show interactive connections between acquired structures of knowledge and cognitive processes (Glaser, 1984; Klahr & Kotovsky, 1989). The results of this newer work have suggested the need for close consideration of specific domains.

According to Piaget, the logical stage (Proportional, Concrete operational, Formal operational) that a child has achieved defines the kind of mental processes available to the child and, thus, basically controls what kinds of specific problem he or she will be able to solve. The particular content of the problems is not central or defining of the child's ability. Application of the structural model of the development of thinking to education initially lead to efforts to teach children to think operationally, sometimes by directly training them on the tasks used to estimate the level of logical development in Piagetian research.

These efforts were largely abandoned as it became increasingly clear that evidence would not support a strict stage theory, because performance on different tasks presumably within the same level of competence could be extremely variable. The strict stage theory position has been modified in a number of Neo-Piagetian theories (Bidell & Fischer; Biggs; Case, 1972). Most developmental psychologists now recognize that specific knowledge in addition to logical competence and/or general mental capacity is required. Considerable effort on the part of some Neo-Piagetians are now directed toward uncovering powerful guiding knowledge schemata that are thought to organize thinking and learning in a particular domain of knowledge.

All three strands of psychological theory, then the differential, the information processing, and the developmental structuralist have come to recognize that both specific knowledge and general competencies are needed to account for the varied performances of individuals. (Bidell & Fischer, 1972). They continue to view that the task of cognitive psychology as building improved accounts of the structure of competence so that , eventually , we will become able to predict performance far better than we do now, as a function of

defined competencies interacting with specific motives and contents.

To deny a fundamental distinction between competence and performance, Dr. Resnick's review of research and practical efforts to teach higher order thinking skills has concluded that shaping a disposition to critical thought is as important in developing higher order cognitive abilities in students as teaching particular skills of reasoning and thinking (Resnick 1987 a).

To apply this line of reasoning to the school of mathematics we as mathematics teachers have to design a new set of cultural practices for the mathematics classroom. We need to create an environment in which children would practice mathematics as a field in which there are open questions and arguments, in which interpretation, reasoning, and debate are all key components of critical thought and play a legitimate and expected role. To do this we need to revise mathematics teaching in the direction of treating mathematics as if it is an ill-structured discipline (Resnick , 1989 b). That is, we need to take seriously, with and for young learners, the propositions that mathematical statements can have more than one

interpretation. That interpretation is the responsibility of every individual using mathematical expressions, and the argument and debate about interpretations and their implications are a normal part of mathematical activity. Participating in such an environment will develop capabilities and skills not only in applying mathematics but also in thinking mathematically . In short it will socialize children into a developmentally appropriate form of the cultural practice of mathematics as a mode of thought, reasoning, and problem solving.

How early can such a program begin? Is it necessary to first teach "basic knowledge" (basic number combinations and arithmetic procedures) before children have anything to reason about ? Is an interpretation- oriented mathematics program suitable for all children or only for the educationally able and socially favored?

These questions will be answered first, with the belief that children entering school already know enough to begin to participate in a reasoning mathematics program. Then the program itself will be described along with some evidence on it's effects and results.

The Intuitive Basics for Early Mathematical Reasoning -

A substantial body of research accumulated over the past decade has suggested that almost all children come to school with a substantial body of knowledge about quantity relations and that children are capable of using this knowledge as a foundation for understanding numbers and arithmetic (Resnick, 1989 a; Resnick & Greeno, 1990, for interpretive views). Knowledge developed prior to school includes understanding of some basic relations involving quantitative properties of objects, along with knowledge of the rules for counting sets of objects.

During the preschool years, children develop a large store of knowledge about how quantities of physical material behave in the world. This knowledge, acquired from manipulating and talking about physical material, allows children to make judgments about comparative amounts and sizes and to reason about changes in amounts and quantities. Before children are two years old, they express quantity judgments in the form of absolute size labels such as big, small, lots, and little (Clark, 1983). They can see two trees and declare one taller than the other, examine two glasses of milk and declare that one contains more than the other. These comparisons are initially based on direct

perceptual judgments without any measurement process. However, they form a basis for eventual numerical comparisons of quantity. Children can be fooled by perceptual cues or language that distracts them from quantity, but they possess a basic understanding of addition, subtraction, and conservation.

Gelman and her colleagues have done a series of research about what it means to understand counting, showing that children as young as three or four years of age implicitly know the key principles that allow counting to serve as a vehicle of quantification (Gelman & Gallistel, 1978). These principles include the knowledge that number names must be matched one-for-one with the objects in a set and that the order of the number names matters, but the order in which the objects are touched does not. Knowledge of these principles is inferred from the ways in which children solve novel counting problems. For example, if asked to make the second object in a row "number 1", children do not neglect the first object entirely but, rather assign it one of the higher number names in the sequence.

Other research has shown that, although children may know all of the principles of counting and be able to use counting to quantify given sets of objects or to create sets of specified sizes, they may not, at a certain point, have fully integrated their counting knowledge with their reasoning (Michie , 1984; Saxe, 1977; Siegler, 1981; Sophian, 1987). This research has also shown that many children who know how to count sets do not spontaneously count in order to compare sets. This means that counting and reasoning exist initially as separate knowledge systems, isolated from each other.

Several researchers (Carpenter & Moser, 1984; DeCorte & Verschaffel, 1987; Nesher, 1982; Riley & Greeno, 1988; Vergnaud, 1982) have shown that children entering school can solve many simple story problems by applying their counting skills to sets they create as they build physical models of the story situations. Because the stories involve the same basic relationships among quantities as their basic knowledge of reasoning, extensive practice in solving problems via counting should help children not only develop their ability to solve problems using exact numerical measures, but also lead them to interpret numbers themselves as the entities that are mentally compared, increased and

decreased, or organized into parts and wholes by the schemata (Resnick & Greeno, 1990).

CHAPTER III

Design Of The Program

To find an answer to the question of how mathematical competence develops in school children, the writer reviewed three different strands of psychological theory; the differential, the informational processing, and the developmental structuralist.

The fact that mathematical concepts and mathematical forms of reasoning are implicated in many domains of cognitive functioning and difficulties in mathematics learning can block access to many educational and career opportunities, motivated the writer to study a new effort set to develop a primary arithmetic teaching method (for grades 1 through 3) by Dr. Lauren Resnick at learning center in University Of Pittsburgh. About 45 children from the first, second, and third grades of an urban area school were chosen for this program. The program has been under development for over two years by a group of mathematics teachers. The school in which they worked served mainly minority (94 percent were African Americans), low income (69 percent were eligible for

free or reduced price lunches) children and located in Eastern part of The United State Of America. The writer will describe the program and evaluate the results of the program along with presenting her opinions and recommendations.

Principles for a Reasoning-Based Arithmetic Program -

In order to provide for children a consistent environment in which they would be socialized to think of themselves as mathematical reasoners and to behave accordingly, we need a program in which children would successfully learn the traditional "basics" of arithmetic calculation as well as more complex forms of reasoning and argumentation and experimentation. This program would be based on a set of six principles that guide our thinking and experimentation.

1. Stimulating the use of counting in the context of the compare, increase/decrease and part/whole schemata through extensive problem-solving practice.

2. Developing children's trust in their own mathematical knowledge. Ask children to explain and justify their procedures for solving problem.

3. Using a standard mathematical notation to record conversations which help children to link their thinking to the formal language of mathematics.

4. Since children know, in non-numerically quantified form, something about properties such as commutativity, associativity, and additive inverse, a major goal of the first year or two of school mathematics is to "mathematize" this knowledge. That is, quantify it and link it to formal expressions and operations.

5. Encouraging children to find problems for themselves that would keep them practicing numbers, facts, and mathematical reasoning. It is important that children come to view mathematics as something that can be found everywhere, not just in school or in problems posed by a teacher.

6. Discussion and argumentation are essential to creating a culture which uses critical thought. To encourage this talk in a typical daily lesson, a single, relatively complex problem is presented on the chalkboard by teacher. The first phase is a class discussion of what the problem means-- what kind of information is given? What possible methods of solution

are there?, and the like. In the second phase, teams of children work together on solving the problem, using drawings, manipulatives, and role playing to support their discussions and solutions. The teams are responsible for developing a solution and explaining why their solution is a mathematically and practically appropriate one. In the third phase of the lesson, teams of students successively present their solutions and justifications to the whole class. The teacher presses for explanations and challenges those that are incomplete or incorrect. In all of these discussions, children are permitted to express themselves in ordinary language. They discuss why several different solutions could all work or why certain ones are better than others.

CHAPTER IV

Results Of The Program

It is little over two years that this program has been under development by some ambitious teachers to improve their teaching. Data gathered from the school's standardized testing program and interviews with some of the children over the course of a year, along with some reports of child and parent reactions to the overall program have shown that this kind of thinking based program succeeds in teaching the basic number facts and arithmetic procedures that are the core of the traditional primary mathematics program (Resnick, 1987b). The program shows that an interpretation-and discussion-oriented mathematics program can begin at the outset of school, by building on the intuitive mathematical knowledge that children have as they enter school. It also shows that teaching facts and skills along with thinking and reasoning both can be developed simultaneously.

Formal evaluation data consists of scores from the California Achievement Test (CAT), which is administered annually in the school each September.

First were tested at the beginning of second grade, second graders at the beginning of third grade, and third graders at the beginning of fourth grade. Figure 1 (see page 29) compares performance of the first graders in the program with a control group --the preceding year's first grade, taught by the same teacher.(page 29).

For each group, mean percentile ranks are shown for the quantitative skill area of the Metropolitan Readiness Test given in March of the kindergarten year and for the mathematics section of the CAT test given in the September following first grade. As can be seen, there was a dramatic positive effect of the program in grade: The mean percentile score rose from 31.3 on the kindergarten test to 84.4 on the post-first grade test; the control group's performance remained flat over the comparable time period. The difference between the groups is highly significant statistically. As important, the whole distribution shifted upward as a result of the program: The lowest scoring program child was 66th percentile; the highest scoring child the preceding year was at the 51st percentile. Thus, the program appeared effective for children of all ability levels.

Figure 2 (see page 30) compares the second grade program group with its control class--the previous

year's second grade, taught by the same teacher.

ANOVAs showed the differences to be highly reliable for both the concepts and applications and the computation subtests. Figure 3 (see page 31) compares the third grade program class with its control, again the preceding year's class taught by the same teacher.

ANOVAs showed strong statistical significance for the concepts and applications subtest, but only marginal significance for the computation subtest. Except for third grade computation, medians as well as means were higher for each group after the program intervention than before, indicating positive effects for children at all levels of ability.

This data tell only part of the story, of course. There is a great deal more that we would like to know for which we do not yet have systematic data. Nevertheless, we can point to some indicators based on the interviews, class observation, and reports from the school. All first graders were interviewed three times during the year, focusing on their knowledge of counting and addition and subtraction facts, along with their methods for calculating and their understanding of the principles of commutativity, compensation, and the complementarity of addition and subtraction. At the outset, these children, as might be expected given their

complementarity of addition and subtraction. At the outset, these children, as might be expected given their socioeconomic status and their parent's generally low educational background, were not highly proficient. Only one third of them could count orally to 100 or beyond, and most were unable to count reliably across decade boundaries. About a third could not solve small number addition problems, even with manipulatives or finger counting and plenty of encouraging support from the interviewer. By December the picture was sharply different. All but a handful of children were performing both addition and subtraction problems successfully, and all of these demonstrated knowledge of the commutativity of addition. At least half were also using invented procedures such as counting on from the larger of two addends or using procedures that showed that they understood principles of complementarity of addition and subtraction. By the end of the school year, essentially all children were performing in this way, and many were successfully solving and explaining multidigit problems.

The following additional evidence indicates that the program was having many of the desired effects. The children displayed multiple examples of confidence

in doing mathematical work. Many sang to themselves as they took the standardized test. When visitors came to the classroom, they would offer to show off by solving math problems. They frequently asked for harder problems. These displays came from children of almost all ability levels. They had not been typical of any except the most able children the preceding year. Homework was more regularly turned in than preceding years, without nagging or pressure from the teacher. Children often asked for extra math periods. Many parents reported that their children loved math and wanted to do math all the time. Parents also sent to school example of problems that children had solved on their own in some everyday family situation. Knowing that the teacher frequently used such problems in class, parents asked that their child's problems be used. It is notable that this kind of parent engagement occurred in a population of parents that is traditionally alienated from the school and tends not to interact with teachers or school officials.

FIGURE 1

Change in Achievement Test Scores for Grade 1.

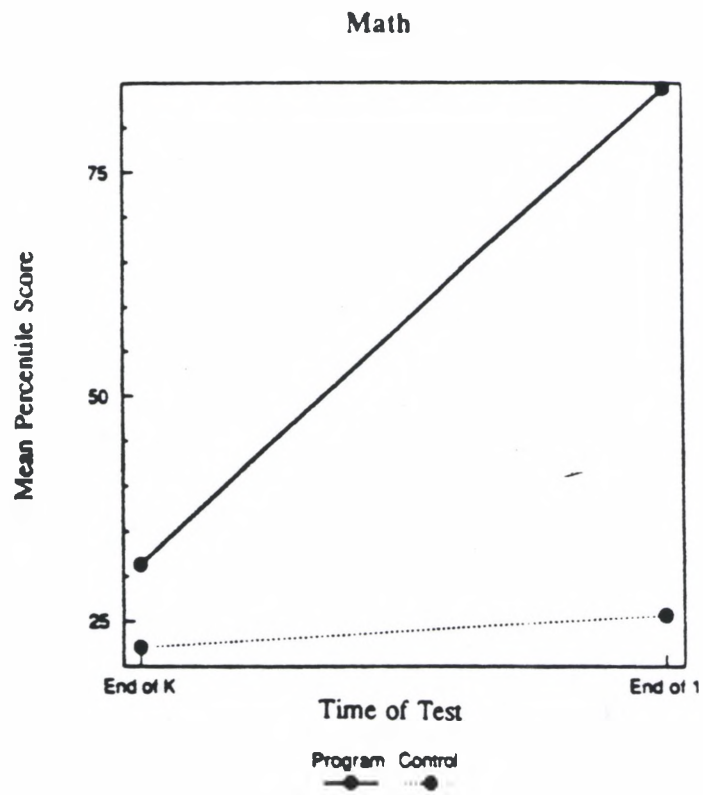


FIGURE 2

Change in Achievement Test Scores for Grade 2.

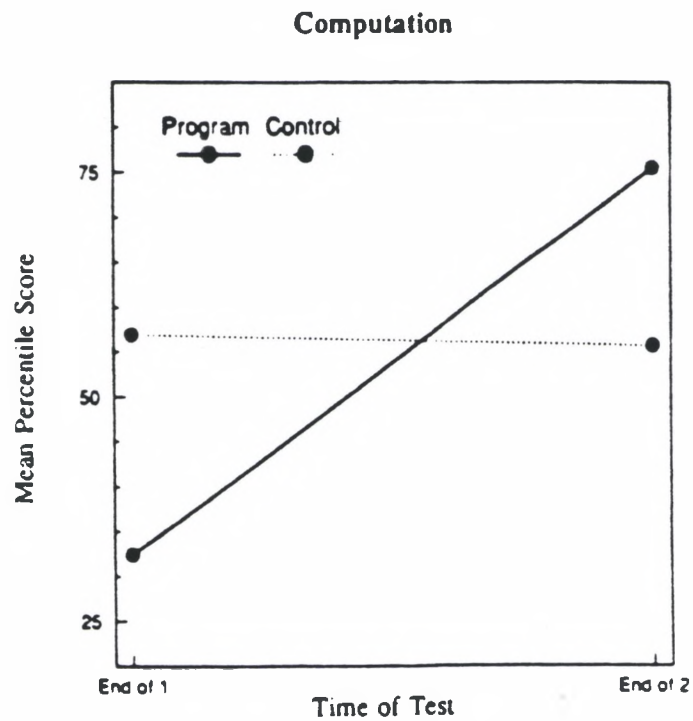
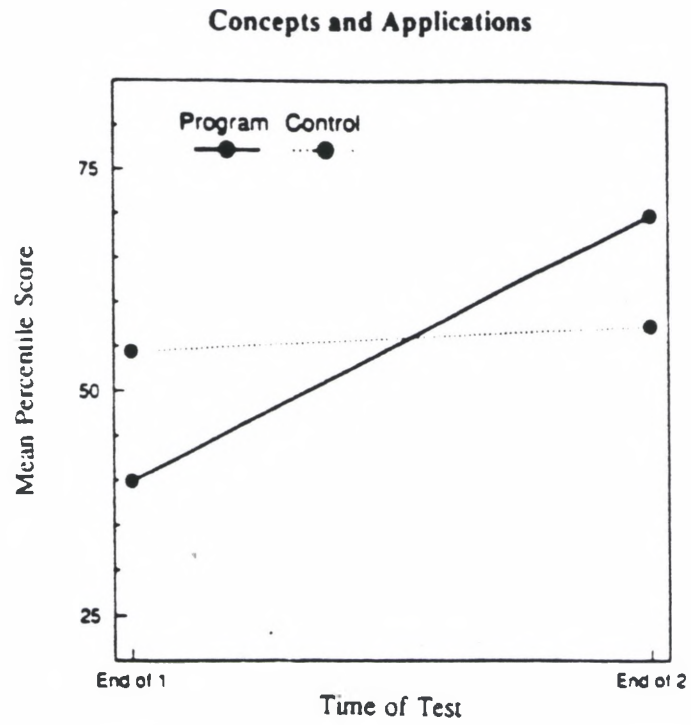
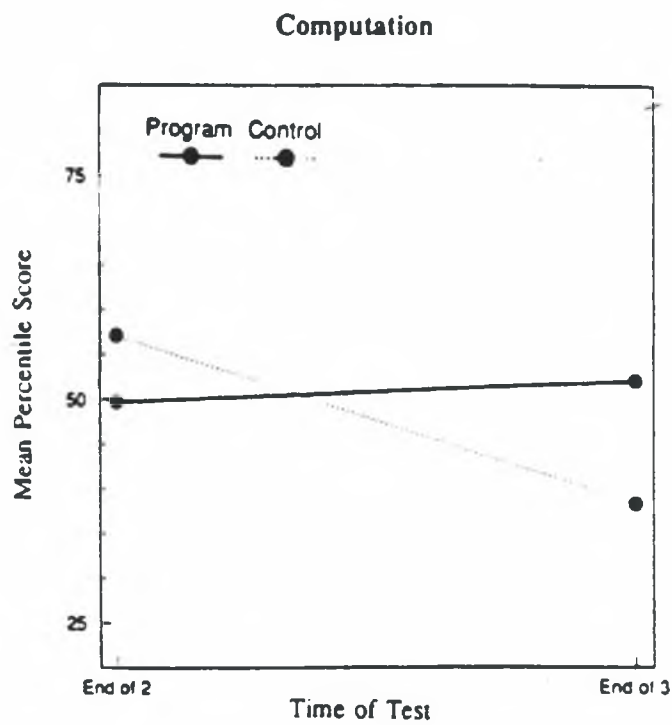
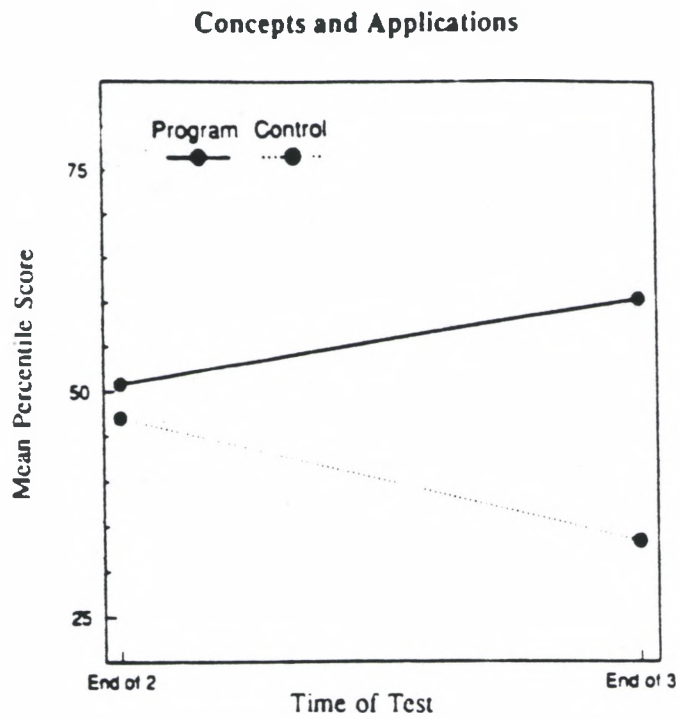


FIGURE 3

Change in Achievement Test Scores for Grade 3.



CHAPTER V

Conclusion

For most people, the main reason for learning mathematics is to acquire tools for solving real, everyday problems. This requires the mental skills of reasoning, problem solving, and analytical thinking. It includes the identification and formulation of specific problems, the solution of a problem translated into a mathematical form, computations, comparison of the results with previous observations, and the drawing of appropriate conclusions

This paper reflects a new theoretical direction about the nature of development, learning, and schooling. This is the view, shared by an increasingly number of thinkers in cognitive science, that human mental functioning must be understood as fundamentally situation specific and context-dependent, rather than as a collection of abstract facts. This research focuses most directly on an interpretation and discussion oriented mathematics program that can begin at the outset of school, by building on the intuitive mathematical knowledge that children have as they en-

ter school. The results of standardized test data show how teachers other than developers succeeds in teaching the basic number facts and arithmetic's procedures. Using this approach also shows, it is not necessary to teach facts and skills first and only then go on to thinking and reasoning. The two can be developed simultaneously.

In short, this project reports on a way of teaching mathematics that helps the learner build on her intuitive math studies so that she sees mathematics learned in school is helpful in the real world.

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